ISSN:0975-9646

Determining an Optimal Parenthesization of a Matrix Chain Product using Dynamic Programming

Vivian Brian Lobo[1], Flevina D'souza[1], Pooja Gharat[1], Edwina Jacob[1], and Jeba Sangeetha Augestin[1]

[1]*Department of Computer Engineering,*
St. Francis Institute of Technology (SFIT)
Mumbai, India 400103

***Abstract*—Dynamic programming is an effective and powerful method for solving a specific class of problems. In computer science, it is used for solving complex problems by breaking a problem into subproblems, solving these subproblems just once, and storing solutions to these subproblems. Matrix chain product is an optimization problem that can be solved through dynamic programming. In this study, we aim to determine an optimal parenthesization of a matrix chain product for a given sequence by dynamic programming using both practical and theoretical approaches.**

***Keywords*—dynamic programming, matrix chain product, optimal solution, parenthesization; sequence**

I. INTRODUCTION

Dynamic programming is one of the sledgehammers of algorithms craft in optimizations, and its usefulness is valued by introduction to various applications [1]. It is an effective and powerful technique for solving a specific class of problems. Dynamic programming is one of the sophisticated algorithm design standards and is a formidable tool that provides classic algorithms for various optimization problems such as shortest path problems, traveling salesman problem, and knapsack problem, including matrix chain product problem [1]. In computer science, it is used for solving complex problems by breaking a problem into subproblems, solving these subproblems just once, and storing solutions to these subproblems. Matrix chain product is a well-known application of optimization problem. It is used in signal processing and network industry for routing [2]. In this study, we aim to determine an optimal parenthesization of a matrix chain product for a given sequence by dynamic programming using both practical and theoretical approaches. Dynamic programming is used when a solution can be recursively described in terms of solutions to subproblems (optimal substructure). An algorithm finds solutions to subproblems and stores them in memory for later use. It is much more efficient than "brute-force methods," which solve the same subproblems frequently [3].

Steps of dynamic programming [4]

1. Characterize the structure of an optimal solution.
2. Recursively define the value of an optimal solution.
3. Compute the value of an optimal solution in a bottom-up fashion.
4. Construct an optimal solution from computed/stored information.

With the help of the abovementioned steps of dynamic programming, we will determine the optimal parenthesization of a matrix chain product using practical as well as theoretical approaches.

The remainder of this paper is organized as follows. Section 2 describes the method that is used for matrix chain product, which includes algorithm to multiply two matrices, multiplication of two matrices, matrix chain product problem, different steps followed under dynamic programming approach, and pseudo code for matrix chain product. Section 3 describes the code for matrix chain product. Section 4 shows the output of matrix chain product. Section 5 explains the theoretical problem solving of matrix chain product. Section 6 shows the complexity of matrix chain product. Finally, section 7 concludes the study.

II. METHOD

Suppose we have a sequence or chain A_1, A_2,…, A_n of n matrices to be multiplied (i.e., we want to compute the product $A_1A_2…A_n$), there are many possible ways (parenthesizations) to compute the product [5].

Example: Consider the chain A_1, A_2, A_3, and A_4 of four matrices. Let us compute the product $A_1A_2A_3A_4$.

There are five possible ways:

1. $(A_1(A_2(A_3A_4)))$
2. $(A_1((A_2A_3)A_4))$
3. $((A_1A_2)(A_3A_4))$
4. $((A_1(A_2A_3))A_4)$
5. $(((A_1A_2)A_3)A_4)$

To compute the number of scalar multiplications, we must know

1. Algorithm to multiply two matrices
2. Matrix dimensions

A. Algorithm to Multiply Two Matrices [6]

Input: Matrices $A_{p\times q}$ and $B_{q\times r}$ (with dimensions $p \times q$ and $q \times r$)
Result: Matrix $C_{p\times r}$ resulting from the product $A \cdot B$

MATRIX-MULTIPLY($A_{p\times q}$, $B_{q\times r}$)

1. for $i \leftarrow 1$ to p
2. for $j \leftarrow 1$ to r
3. $C[i, j] \leftarrow 0$
4. for $k \leftarrow 1$ to q
5. $C[i, j] \leftarrow C[i, j] + A[i, k] \cdot B[k, j]$
6. return C

Scalar multiplication in line 5 dominates time to compute C number of scalar multiplications = pqr

B. *Multiplication of Two Matrices* [7]

MATRIX-MULTIPLY(A, B)

1. if A.columns ≠ B.rows
2. error "incompatible dimensions"
3. else let C be a new A.rows × B.columns matrix
4. for i = 1 to A.rows
5. for j = 1 to B.columns
6. $c_{ij} = 0$
7. for k = 1 to A.columns
8. $c_{ij} = c_{ij} + a_{ik} \bullet b_{kj}$
9. return C

Example: Consider three matrices $A_{10\times100}$, $B_{100\times5}$, and $C_{5\times50}$

There are two ways to parenthesize $((AB)C) = D_{10\times5} \cdot C_{5\times50}$

AB $\Rightarrow 10 \times 100 \times 5 = 5000$ scalar multiplications

DC $\Rightarrow 10 \times 5 \times 50 = 2500$ scalar multiplications } Total: 7500

$(A(BC)) = A_{10\times100} \cdot E_{100\times50}$

BC $\Rightarrow 100 \times 5 \times 50 = 25000$ scalar multiplications

AE $\Rightarrow 10 \times 100 \times 50 = 50000$ scalar multiplications } Total: 7500

C. *Matrix Chain Product Problem*

Given a chain $A_1, A_2, \ldots, A_n$ of n matrices, where for i = 1, 2,…, n, matrix A_i has dimension $p_{i-1} \times p_i$

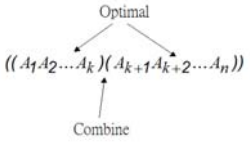
Parenthesize the product $A_1A_2\ldots A_n$ such that the total number of scalar multiplications is minimized [6].

Counting the number of parenthesizations

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \\ \sum_{k=1}^{n-1} P(k)\,p(n-k) & \text{if } n \geq 2 \end{cases}$$

D. *Dynamic Programming Approach* [8]

Step 1: Structure of an optimal parenthesization

1. Let us use the notation $A_{i..j}$ for the matrix that results from the product $A_i\, A_{i+1} \ldots A_j$
2. An optimal parenthesization of the product $A_1A_2\ldots A_n$ splits the product between A_k and A_{k+1} for some integer k where $1 \leq k < n$
3. First compute matrices $A_{1..k}$ and $A_{k+1..n}$; then multiply them to obtain the final matrix $A_{1..n}$
4. *Key observation*: Parenthesizations of subchains $A_1A_2\ldots A_k$ and $A_{k+1}A_{k+2}\ldots A_n$ must also be optimal if the parenthesization of chain $A_1A_2\ldots A_n$ is optimal.
5. In other words, the optimal solution to a problem contains within it the optimal solution to subproblems.
6. 

Step 2: Recursive solution [9]

1. Let $m[i, j]$ be the minimum number of scalar multiplications that are needed to compute $A_{i..j}$
2. Minimum cost to compute $A_{1..n}$ is $m[1, n]$
3. Suppose the optimal parenthesization of $A_{i..j}$ splits the product between A_k and A_{k+1} for some integer k where $i \leq k < j$
4. $A_{i..j} = (A_i\, A_{i+1}\ldots A_k)\cdot(A_{k+1}A_{k+2}\ldots A_j) = A_{i..k} \cdot A_{k+1..j}$
5. Cost of computing $A_{i..j}$ = cost of computing $A_{i..k}$ + cost of computing $A_{k+1..j}$ + cost of multiplying $A_{i..k}$ and $A_{k+1..j}$
6. Cost of multiplying $A_{i..k}$ and $A_{k+1..j}$ is $p_{i-1}p_k p_j$
7. $m[i, j] = m[i, k] + m[k+1, j] + p_{i-1}p_k p_j$ for $i \leq k < j$
8. $m[i, i] = 0$ for $i = 1,2,\ldots,n$
9. But optimal parenthesization occurs at one value of k among all possible $i \leq k < j$
10. Check all these and select the best one.

$$m[i, j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \leq k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_k p_j\} & \text{if } i<j \end{cases}$$

Step 3: Computing the optimal cost [10]

1. To keep track of how to construct an optimal solution, we use a split table s
2. $s[i, j]$ = value of k at which $A_i\, A_{i+1} \ldots A_j$ is split for optimal parenthesization

3. Algorithm

- First compute costs for chains of length $l = 1$
- Then for chains of length $l = 2,3,\ldots$ and so on
- Compute the optimal cost in a bottom-up fashion

Step 4: Constructing an optimal solution

1. The algorithm computes the minimum cost table *m* and split table *s*
2. The optimal solution can be constructed from the split table *s*
3. Each entry $s[i, j] = k$ shows where to split the product $A_i A_{i+1} \ldots A_j$ for the minimum cost.

E. Pseudo Code [4]

The pseudo code for matrix chain product is as follows [4]:
Input: Array *p[0…n]* containing matrix dimensions

Result: Minimum cost table *m* and split table *s*

MATRIX-CHAIN-ORDER(p)

```
1.  n = p.length − 1
2.  let m[1..n, 1..n] and s[1..n − 1, 2..n] be new tables
3.  for i = 1 to n
4.        m[i, i] = 0
5.  for l = 2 to n                    // l is the chain length
6.        for i = 1 to n – l + 1
7.            j = i + l − 1
8.            m[i, j] = ∞
9.            for k = i to j − 1
10.               q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
11.                     If q < m[i, j]
12.                m[i, j] = q
13.                s[i, j] = k
14. return m and s
```

The pseudo code for printing the optimal parenthesization of a matrix chain product is as follows [4]:

PRINT-OPTIMAL-PARENS(s, i, j)

```
1. if i == j
2. print "A" i
3. else print "("
4. PRINT-OPTIMAL-PARENS(s, i, s[i, j])
5. PRINT-OPTIMAL-PARENS(s, s[i, j] + 1, j)
6. print ")"
```

III. PROGRAM CODE FOR MATRIX CHAIN PRODUCT

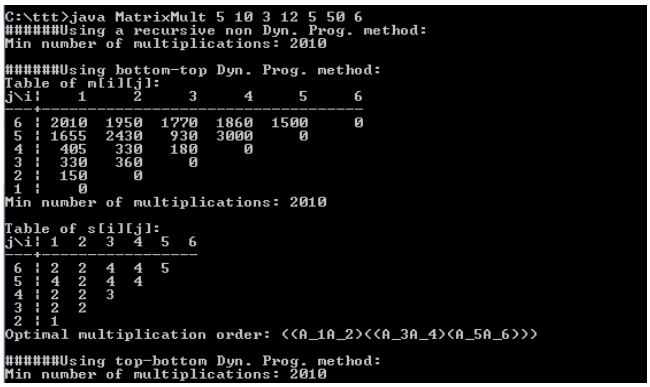
The program code for matrix chain product is as follows:

```
1.  public class MatrixMult
2.  {
3.  public static int[][] m;
4.  public static int[][] s;

5.  public static void main(String[] args)
6.  {
7.  int[] p = getMatrixSizes(args);
8.  int n = p.length-1;
9.  if (n < 2 || n > 15)
10. {
11. System.out.println("Wrong input");
12. System.exit(0);
13. }

14. System.out.println("######Using a recursive non Dyn. Prog. method:");
15. int mm = RMC(p, 1, n);
16. System.out.println("Min number of multiplications: " + mm + "\n");
17. System.out.println("######Using bottom-top Dyn. Prog. method:");
18. MCO(p);
19. System.out.println("Table of m[i][j]:");
20. System.out.print("j\\i|");
21. for (int i=1; i<=n; i++)
22. System.out.printf("%5d ", i);
23. System.out.print("\n---+");
24. for (int i=1; i<=6*n-1; i++)
25. System.out.print("-");
26. System.out.println();
27. for (int j=n; j>=1; j--)
28. {
29. System.out.print(" " + j + " |");
30. for (int i=1; i<=j; i++)
31. System.out.printf("%5d ", m[i][j]);
32. System.out.println();
33. }
34. System.out.println("Min number of multiplications: " + m[1][n] + "\n");
35. System.out.println("Table of s[i][j]:");
36. System.out.print("j\\i|");
37. for (int i=1; i<=n; i++)
38. System.out.printf("%2d ", i);
39. System.out.print("\n---+");
40. for (int i=1; i<=3*n-1; i++)
41. System.out.print("-");
42. System.out.println();
43. for (int j=n; j>=2; j--)
44. {
45. System.out.print(" " + j + " |");
46. for (int i=1; i<=j-1; i++)
47. System.out.printf("%2d ", s[i][j]);
48. System.out.println();
49. }
50. System.out.print("Optimal multiplication order: ");
51. MCM(s, 1, n);
52. System.out.println("\n");
```

```
53. System.out.println("######Using top-bottom Dyn. Prog. method:");
54. mm = MMC(p);
55. System.out.println("Min number of multiplications: " + mm);
56. }

57. public static int RMC(int[] p, int i, int j)
58. {
59. if (i == j) return(0);
60. int m_ij = Integer.MAX_VALUE;
61. for (int k=i; k<j; k++)
62. {
63. int q = RMC(p, i, k) + RMC(p, k+1, j) + p[i-1]*p[k]*p[j];
64. if (q < m_ij)
65. m_ij = q;
66. }
67. return(m_ij);
68. }

69. public static void MCO(int[] p)
70. {
71. int n = p.length-1;    // # of matrices in the product
72. m = new int[n+1][n+1];    // create and automatically initialize array m
73. s = new int[n+1][n+1];

74. for (int l=2; l<=n; l++)
75. {
76. for (int i=1; i<=n-l+1; i++)
77. {
78. int j=i+l-1;
79. m[i][j] = Integer.MAX_VALUE;

80. for (int k=i; k<=j-1; k++)
81. {
82. int q = m[i][k] + m[k+1][j] + p[i-1]*p[k]*p[j];
83. if (q < m[i][j])
84. {
85. m[i][j] = q;
86. s[i][j] = k;
87. }
88. }
89. }
90. }
91. }

92. public static void MCM(int[][] s, int i, int j)
93. {
94. if (i == j) System.out.print("A_" + i);
95. else
96. {
97. System.out.print("(");
98. MCM(s, i, s[i][j]);
99. MCM(s, s[i][j]+1, j);
100. System.out.print(")");
101. }
102. }

103. public static int MMC(int[] p)
104. {
105. int n = p.length-1;
106. m = new int[n+1][n+1];
107. for (int i=0; i<=n; i++)
108. for (int j=i; j<=n; j++)
109. m[i][j] = Integer.MAX_VALUE;
110. return(LC(p, 1, n));
111. }

112. public static int LC(int[] p, int i, int j)
113. {
114. if (m[i][j] < Integer.MAX_VALUE) return(m[i][j]);

115. if (i == j) m[i][j] = 0;
116. else
117. {
118. for (int k=i; k<j; k++)
119. {
120. int q = LC(p, i, k) + LC(p, k+1, j) + p[i-1]*p[k]*p[j];
121. if (q < m[i][j])
122. m[i][j] = q;
123. }
124. }
125. return(m[i][j]);
126. }


127. public static int[] getMatrixSizes(String[] ss)
128. {
129. int k = ss.length;
130. if (k == 0)
131. {
132. System.out.println("No matrix dimensions entered");
133. System.exit(0);
134. }
135. int[] p = new int[k];
136. for (int i=0; i<k; i++)
137. {
138. try
139. {
140. p[i] = Integer.parseInt(ss[i]);
141. if (p[i] <= 0)
142. {
143. System.out.println("Illegal input number " + k);
144. System.exit(0);
145. }
146. }
147. catch(NumberFormatException e)
148. {
149. System.out.println("Illegal input token " + ss[i]);
150. System.exit(0);
151. }
152. }
153. return(p);
154. }
155. }
```

IV. OUTPUT OF MATRIX CHAIN PRODUCT

The output of matrix chain product is as follows:



```
C:\ttt>java MatrixMult 5 10 3 12 5 50 6
######Using a recursive non Dyn. Prog. method:
Min number of multiplications: 2010

######Using bottom-top Dyn. Prog. method:
Table of m[i][j]:
j\i|    1     2     3     4     5     6
---+------------------------------------
 6 | 2010  1950  1770  1860  1500     0
 5 | 1655  2430   930  3000     0
 4 |  405   330   180     0
 3 |  330   360     0
 2 |  150     0
 1 |    0
Min number of multiplications: 2010

Table of s[i][j]:
j\i| 1  2  3  4  5  6
---+-----------------
 6 | 2  2  4  4  5
 5 | 4  2  4  4
 4 | 2  2  3
 3 | 2  2
 2 | 1
Optimal multiplication order: ((A_1A_2)((A_3A_4)(A_5A_6)))

######Using top-bottom Dyn. Prog. method:
Min number of multiplications: 2010
```

The abovementioned program code for matrix chain product was written in notepad and compiled and successfully executed in Java environment using Java Development Kit (jdk) version 8, jdk1.8.0_20-b26 (32 bit). The system configuration is as follows:

Operating system	Windows 7 Home Basic
Processor	Intel(R) Core(TM) i5-2450M CPU @ 2.50GHz 2.50 GHz
RAM	4 GB
System type	64-bit OS

V. THEORETICAL PROBLEM SOLVING OF MATRIX CHAIN PRODUCT

Problem statement: Determine an optimal parenthesization of a matrix chain product using dynamic programming for the given sequence (5, 10, 3, 12, 5, 50, 6)

To determine an optimal parenthesization of a matrix chain product using dynamic programming, we considered a problem with the following sequence (5, 10, 3, 12, 5, 50, 6). The solution to this problem is explained below.

Step 0:

Consider $P_0 = 5$, $P_1 = 10$, $P_2 = 3$, $P_3 = 12$, $P_4 = 5$, $P_5 = 50$, $P_6 = 6$

$m[1, 1] = 0$, $m[2, 2] = 0$, $m[3, 3] = 0$, $m[4, 4] = 0$, $m[5, 5] = 0$, $m[6, 6] = 0$

Step 1:

$$m[i, j] = m[i, k] + m[k + 1, j] + p_{i-1}\, p_k\, p_j$$

where $k = j - 1$

$m[1, 2] = m[1, 1] + m[2, 2] + (P_0 \times P_1 \times P_2)$
$= 0 + 0 + (5 \times 10 \times 3) = 150$

$m[2, 3] = m[2, 2] + m[3, 3] + (P_1 \times P_2 \times P_3)$
$= 0 + 0 + (10 \times 3 \times 12) = 360$

$m[3, 4] = m[3, 3] + m[4, 4] + (P_2 \times P_3 \times P_4)$
$= 0 + 0 + (3 \times 12 \times 5) = 180$

$m[4, 5] = m[4, 4] + m[5, 5] + (P_3 \times P_4 \times P_5)$
$= 0 + 0 + (12 \times 5 \times 50) = 3000$

$m[5, 6] = m[5, 5] + m[6, 6] + (P_4 \times P_5 \times P_6)$
$= 0 + 0 + (5 \times 50 \times 6) = 1500$

Step 2:

$$m[i, j] = m[i, k] + m[k + 1, j] + p_{i-1}\, p_k\, p_j$$

where $k = j - 1$

$m[1, 3] = m[1, 1] + m[2, 3] + (P_0 \times P_1 \times P_3)$
$= 0 + 360 + (5 \times 10 \times 12) = 600$

$m[1, 3] = m[1, 2] + m[3, 3] + (P_0 \times P_2 \times P_3)$
$= 150 + 0 + (5 \times 3 \times 12) =$ **330**

$m[2, 4] = m[2, 2] + m[3, 4] + (P_1 \times P_2 \times P_4)$
$= 0 + 180 + (10 \times 3 \times 5) =$ **330**

$m[2, 4] = m[2, 3] + m[4, 4] + (P_1 \times P_3 \times P_4)$
$= 360 + 0 + (10 \times 12 \times 5) = 960$

$m[3, 5] = m[3, 3] + m[4, 5] + (P_2 \times P_3 \times P_5)$
$= 0 + 3000 + (3 \times 12 \times 50) = 4800$

$m[3, 5] = m[3, 4] + m[5, 5] + (P_2 \times P_4 \times P_5)$
$= 180 + 0 + (3 \times 5 \times 50) =$ **930**

$m[4, 6] = m[4, 4] + m[5, 6] + (P_3 \times P_4 \times P_6)$
$= 0 + 1500 + (15 \times 5 \times 6) =$ **1860**

$m[4, 6] = m[4, 5] + m[6, 6] + (P_3 \times P_5 \times P_6)$
$= 3000 + 0 + (12 \times 50 \times 6) = 6600$

Step 3:

$$m[i, j] = m[i, k] + m[k + 1, j] + p_{i-1}\, p_k\, p_j$$

where $k = j - 1$

$m[1, 4] = m[1, 1] + m[2, 4] + (P_0 \times P_1 \times P_4)$
$= 0 + 330 + (5 \times 10 \times 5) = 580$

$m[1, 4] = m[1, 2] + m[3, 4] + (P_0 \times P_2 \times P_4)$
$= 150 + 180 + (5 \times 3 \times 5) =$ **405**

$m[1, 4] = m[1, 3] + m[4, 4] + (P_0 \times P_3 \times P_4)$
$= 330 + 0 + (5 \times 12 \times 5) = 630$

$m[2, 5] = m[2, 2] + m[3, 5] + (P_1 \times P_2 \times P_5)$
$= 0 + 930 + (10 \times 3 \times 50) =$ **2430**

$m[2, 5] = m[2, 3] + m[4, 5] + (P_1 \times P_3 \times P_5)$
$= 360 + 3000 + (10 \times 12 \times 50) = 9360$

$m[2, 5] = m[2, 4] + m[5, 5] + (P_1 \times P_4 \times P_5)$
$= 330 + 0 + (10 \times 5 \times 50) = 2830$

$m[3, 6] = m[3, 3] + m[4, 6] + (P_2 \times P_3 \times P_6)$

$= 0 + 1860 + (3 \times 12 \times 6) = 2076$

$m[3, 6] = m[3, 4] + m[5, 6] + (P_2 \times P_4 \times P_6)$
$= 180 + 1500 + (3 \times 5 \times 6) =$ **1770**

$m[3, 6] = m[3, 5] + m[6, 6] + (P_2 \times P_5 \times P_6)$
$= 930 + 0 + (3 \times 50 \times 6) = 1830$

Step 4:

$m[i, j] = m[i, k] + m[k + 1, j] + p_{i-1}\, p_k\, p_j$

where $k = j - 1$

$m[1, 5] = m[1, 1] + m[2, 5] + (P_0 \times P_1 \times P_5)$
$= 0 + 2430 + (5 \times 10 \times 50) = 4930$

$m[1, 5] = m[1, 2] + m[3, 5] + (P_0 \times P_2 \times P_5)$
$= 150 + 930 + (5 \times 3 \times 50) = 1830$

$m[1, 5] = m[1, 3] + m[4, 5] + (P_0 \times P_3 \times P_5)$
$= 330 + 3000 + (5 \times 12 \times 50) = 6330$

$m[1, 5] = m[1, 4] + m[5, 5] + (P_0 \times P_4 \times P_5)$
$= 405 + 0 + (5 \times 5 \times 50) =$ **1655**

$m[2, 6] = m[2 ,2] + m[3, 6] + (P_1 \times P_2 \times P_6)$
$= 0 + 1770 + (10 \times 3 \times 6) =$ **1950**

$m[2, 6] = m[2, 3] + m[4, 6] + (P_1 \times P_3 \times P_6)$
$= 360 + 1860 + (10 \times 12 \times 6) = 2940$

$m[2, 6] = m[2, 4] + m[5, 6] + (P_1 \times P_4 \times P_6)$
$= 330 + 1500 + (10 \times 5 \times 6) = 2130$

$m[2, 6] = m[2, 5] + m[6, 6] + (P_1 \times P_5 \times P_6)$
$= 2430 + 0 + (10 \times 50 \times 6) = 5430$

Step 5:

$m[i, j] = m[i, k] + m[k + 1, j] + p_{i-1}\, p_k\, p_j$

where $k = j - 1$

$m[1, 6] = m[1, 1] + m[2, 6] + (P_0 \times P_1 \times P_6)$
$= 0 + 1950 + (5 \times 10 \times 6) = 2250$

$m[1, 6] = m[1, 2] + m[3, 6] + (P_0 \times P_2 \times P_6)$
$= 150 + 1770 + (5 \times 3 \times 6) =$ **2010**

$m[1, 6] = m[1, 3] + m[4, 6] + (P_0 \times P_3 \times P_6)$
$= 330 + 1860 + (5 \times 12 \times 6) = 2550$

$m[1, 6] = m[1, 4] + m[5, 6] + (P_0 \times P_4 \times P_6)$
$= 405 + 1500 + (5 \times 5 \times 6) = 2055$

$m[1, 6] = m[1, 5] + m[6, 6] + (P_0 \times P_5 \times P_6)$
$= 1655 + 0 + (5 \times 50 \times 6) = 3155$

The optimal parenthesization of a matrix chain product using dynamic programming for the given sequence (5, 10, 3, 12, 5, 50, 6) is $((A_1 \times A_2)((A_3 \times A_4)(A_5 \times A_6)))$. From the above solution of the given problem, we can see that all possible ways of obtaining the parenthesization of a matrix chain product using dynamic programming are performed. In other words, all possible solutions are obtained, and from those solutions, the optimal solution is taken, i.e., from Step 2 to Step 5, we have selected only those solutions that provide the least or minimum value, which can be reflected in the minimum cost table, as shown in Fig. 1. The respective *k* values are included in the split table, as shown in Fig. 2.

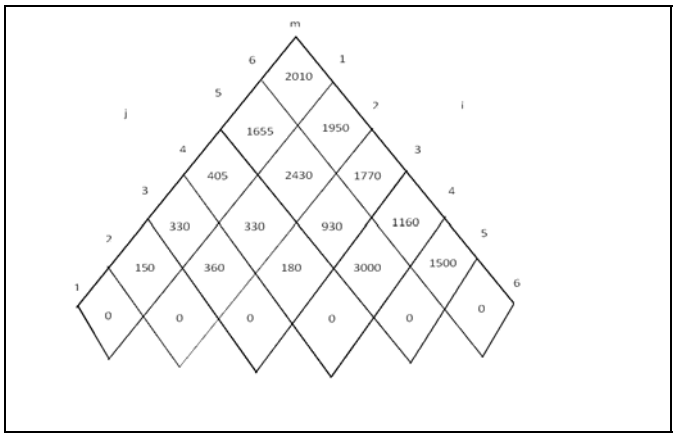


Fig. 1. Minimum cost table

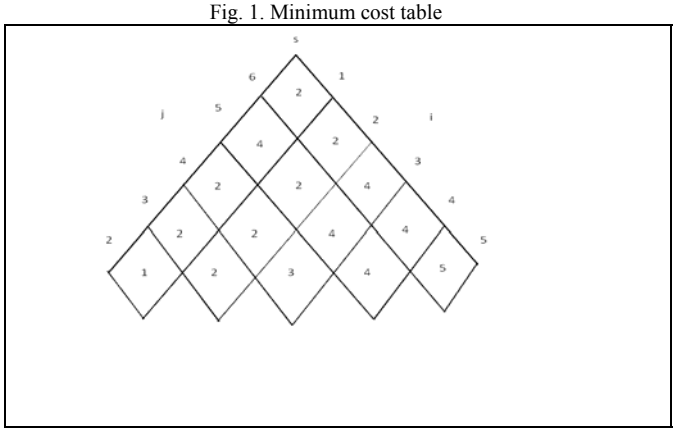


Fig. 2. Split table

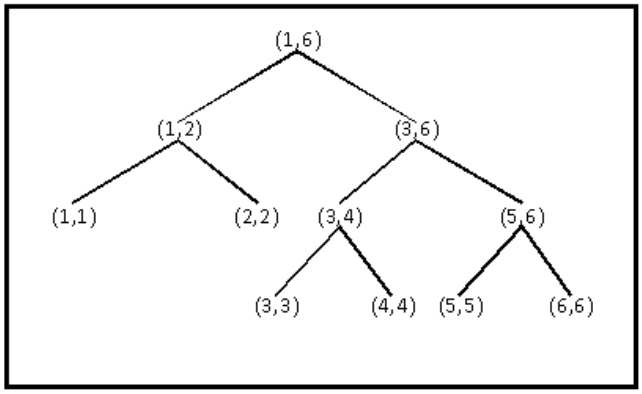


Fig. 3. Tree for optimal parenthesization

Backtracking is a method that helps in determining the optimal parenthesization of a matrix chain product for a given sequence by dynamic programming (i.e., it helps in obtaining the final solution, as shown below). In Fig. 3, we observe that the leaf nodes in the tree for optimal parenthesization are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), and (6, 6). However, to obtain these leaf nodes, we first check Step 5. In Step 5, the minimum value obtained is **2010**, which is derived from m[1, 6], which is a combination of m[1, 2] and m[3, 6]. Hence, we consider (1, 6) as the first coordinate (Fig. 3). Now, we see that the value of m[i, j] in m[1, 6] is m[1, 2] and the value of m[k + 1, j] in m[1, 6] is m[3, 6], and thus, we check for m[1, 2] and m[3, 6] from Step 1 to 4. The desired value of m[1, 2] is found in Step 1 and that of m[3, 6] is found in Step 3. We observe that m[1, 2] has a single value (i.e., it does not have the concept of minimum values), and so, we observe that m[1, 2] is a combination of m[1, 1] and m[2, 2]. Thus, we can split (1, 2) as (1, 1) and (2, 2), as shown in Fig. 3. Now, for m[3, 6], we check in which step does it occur and we consider the minimum value. From our observation, we perceive that m[3, 6] is present in Step 3 and the minimum value is **1770**. Furthermore, m[3, 6] is a combination of m[3, 4] and m[5, 6]. Thus, we can split (3, 6) as (3, 4) and (5, 6), which can be seen in Fig. 3. Finally, we check for m[3, 4] and m[5, 6]. The abovementioned procedure is followed and (3, 4) is split as (3, 3) and (4, 4), whereas (5, 6) is split as (5, 5) and (6, 6) (Fig. 3). We stop when the leaf nodes are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), and (6, 6). From Fig. 3, we can now determine the optimal solution. First, we obtain $(A_1 \times A_2)$. Second, we obtain $(A_3 \times A_4)$ and $(A_5 \times A_6)$. Third, we combine $((A_3 \times A_4)(A_5 \times A_6))$, and finally, we combine $((A_1 \times A_2)((A_3 \times A_4)(A_5 \times A_6)))$, which gives the final solution.

VI. COMPLEXITY OF MATRIX CHAIN PRODUCT

The time complexity of matrix chain product is $O(n^3)$, and the space complexity of matrix chain product is $O(n^2)$ [10].

VII. CONCLUSION

Matrix chain product problem encompasses the question how the optimal classification for performing a series of operations can be determined. Moreover, matrix chain product problem is not actually to perform multiplication but simply to decide the order to perform multiplication. Thus, we have successfully determined the optimal parenthesization of a matrix chain product for a given sequence by dynamic programming using practical as well as theoretical approaches.

REFERENCES

[1] B. Bhowmik, "Simplified optimal parenthesization scheme for matrix chain multiplication problem using bottom-up practice in 2-tree structure," Journal of Applied Computer Science & Mathematics, vol. 11, no. 5, pp. 9-14, 2011.

[2] R. Lakhotia, S. Kumar, R. Sood, H. Singh, and J. Nabi, "Matrix-chain multiplication using greedy and divide-conquer approach," International Journal of Computer Trends and Technology, vol. 23, no. 2, pp. 65-72, May 2015.

[3] https://edurev.in/studytube/10202014-1--/5dd4be9f-8f66-40ec-b5d9-c99adae64fc4_p

[4] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, "Introduction to algorithms," MIT Press, July 14, 1990.

[5] www.slidefinder.net/m/matrix_mult/matrix-mult/7256761

[6] http://pt.slideshare.net/kumar_vic/matrix-mult-class17

[7] http://www.purplemath.com/modules/mtrxmult.htm

[8] http://docslide.us/documents/analysis-of-algorithms-chapter-07-dynamic-programming.html

[9] P. Gupta, V. Agarwal, and M. Varshney, "Design and analysis of algorithms," 2nd Edition, PHI Learning Private Limited (New Delhi), ISBN-978-81-203-4663-5.

[10] http://www.personal.kent.edu/~rmuhamma/Algorithms/MyAlgorithms/Dynamic/chainMatrixMult.htm